The Existence and Uniqueness of Solutions to The Riemann-Liouville Fractional Iterative System

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Abstract: This paper is devoted to study the iterative problem of Riemann-Liouville fractional derivative, give the Green's function of the boundary value problem, and introduce the Gronwall inequality. We consider an initial value problem for a Riemann-Liouville fractional derivative equation. The appliance utilized in this work, is the fixed point theorem of Leray-Schauder and the Gronwall inequality.

1. Introduction

It is well established that the fractional differential equation has always been the hot topic in mathematical research. In 2018, Cheng et al [1] established the precise controllability of the fractional order control system with time-varying delay. In [8], according to the Krasnosel' skii's fixed point theorem, X.Zhang et al obtained the solutions for the following system

$$\begin{cases} -D_{0^+}^{p}y(t)=p(t)f(t,y(t))-q(t), \ t\in(0,1) \\ y(0)=y'(0)=0, \ y(1)=0, \end{cases} \tag{1}$$

where D^p denotes the Riemann-Liouville fractional derivative. Similar fractional order equation of form

$$\begin{cases} -D^{p}y(t)+p(t)f(t,y(t))+q(t)=0, \ t\in(0,1) \\ y(0)=y'(0)=0, \ y(1)=0, \end{cases} \tag{2}$$

has been studied by Yujun Cui in [2] and the author obtained the characteristics of solution to the system. Paul W. Eloe and Tyler Masthay [3] studied an initial value problem of nonlinear Caputo system with the order of α, where $0<\alpha<1$, and developed some fundamental results. The iterative problems has been of considerable interest to mathematical community in recent years. Petuhov [6] studied the second-order iterative equation

$$l''=\lambda l(l(t)), \tag{3}$$

where $l(t):[-T,T]\to[-T,T]$, and $l(0)=l(T)=\alpha$, under the conditions on λ and α. And then in [5], Wang considered the existence of the problem

$$\lambda'=f(\lambda(\lambda(t))), \lambda(a)=a, \tag{4}$$

where a is the end point of an interval. In 2018, Kaufamann [4] further obtained the existence and uniqueness of the solutions to the second-order iterative boundary-value equation

$$\lambda''=f\left(t,\lambda(t),\lambda^{[2]}(t)\right), \tag{5}$$

where $\lambda^{[2]}(t)=\lambda(\lambda(t))$, with the conditions $\lambda(\rho)=\rho, \ \lambda(\gamma)=\gamma$ or $\lambda(\rho)=\gamma, \ \lambda(\gamma)=\rho$. Surprisingly little attention has been devoted into the fractional order iterative systems. Motivated by the above works, we consider the following iterative equations:

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$$D^{\alpha}x(t)= f\left(t, x(t), x^{[2]}(t)\right), \rho \le t \le \gamma, n-1<\alpha<n \tag{6}$$

Where $x^{[2]}(t)$ means $x(x(t))$, D^{α} denotes the Riemann-Liouville fractional derivative. First, we introduce a definition and a lemma.

Definition 1.1. Define f to be in the interval (ρ,γ), for any complex number $\alpha>0$, we have the Riemann-Liouville fractional integral

$$D^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)}\int_{\rho}^{t} f(\theta)(t-\theta)^{\alpha-1}d\theta. \tag{7}$$

Lemma 1.1. (see [7]) Suppose $\alpha>0$, $s(t)\in[0,T)$ is locally integrable, and it is a nondecreasing and nonnegative function. Let Q_{α} be the Mittag-Leffler function. For $k(t)\in C[0,T)$ which is nonnegative, nondecreasing and continuous fucntion. Let $\omega(t)$ be a nonnegative and locally function in $C[0,T)$, which satisfying

$$\omega(t)\le s(t)+k(t)\int_{a}^{t}(t-\tau)^{\alpha-1}\omega(\tau)d\tau. \tag{8}$$

Then

$$\omega(t)\le s(t)Q_{\alpha}\left(k(t)\Gamma(\alpha)t^{\alpha}\right), \tag{9}$$

where $Q_{\alpha}(z)=\sum_{i=0}^{\infty}\frac{z^{i}}{\Gamma(i\alpha+1)}$.

Then we need the assumptions on the function of f as follows

(H1) If there exist positive numbers N,Q ,which satisfying the inequality $-K< f(t,\mu,\eta)<Q$, for all $t\in[\rho,\gamma]$, $\mu,\eta\in R$.

(H2) Suppose that the constants $M, F>0$, and satisfied

$$\left(f(t,w_1,r_1)-f(t,w_2,r_2)\right)(w_1-w_2)\le M|w_1-w_2|^2+F|r_1-r_2|^2, \tag{10}$$

for all $t\in[\rho,\gamma]$, $w_1,w_2,r_1,r_2\in R$.

Now, we introduce the main result in this article.

Theorem 1.1. *For given the continuous function f, if it satisfies the assumptions (H1),(H2), then there exists a unique solution for the fractional system (1).*

Proof of Theorem 1.1.

Consider the boundary value problem for the following Riemann-Liouville fractional integral equation

$$\begin{cases} D^{\alpha}x(t)= f\left(t, x(t), x^{[2]}(t)\right), \ u<t<v, \alpha>0 \\ x^{(j)}(t)=0, \ x(u)=u>0, \ x(v)=v \end{cases} \tag{11}$$

where $x^{[2]}(t)=x(x(t))$, $j\in(0,n-2)$. The solution of $J^{\alpha}D^{\alpha}x(t)=0$ is

$x(t)=\beta_1 t^{\alpha-1}+\beta_2 t^{\alpha-1}+\cdots+\beta_n t^{\alpha-n}$, where $\beta_i\in R, i=1,2,\cdots,n.$

We can deduce that, the solution is equivalent to

$$x(t)=\beta_1 t^{\alpha-1}+\beta_2 t^{\alpha-2}+\cdots+\beta_n t^{\alpha-n}+\frac{1}{\Gamma(\alpha)}\int_{\rho}^{t}(t-\theta)^{\alpha-1} f\left(\theta, x(\theta), x^{[2]}(\theta)\right)d\theta. \tag{12}$$

By differentials,

$$x(t)=\beta_1 t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}(t-\theta)^{\alpha-1} f\left(\theta, x(\theta), x^{[2]}(\theta)\right)d\theta. \tag{13}$$

According to $x(\gamma)=\gamma$, it follows that

$$\gamma=\beta_1\gamma^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_\rho^\gamma(\gamma-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta, \tag{14}$$

then we have

$$\beta_1=\gamma^{2-\alpha}+\frac{t^{\alpha-1}}{\gamma^{\alpha-1}\Gamma(\alpha)}\int_\rho^\gamma(\gamma-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta. \tag{15}$$

Substitute β_1 into (3), we obtain that

$$\begin{aligned}x(t)&=\gamma^{2-\alpha}t^{\alpha-1}-\frac{t^{\alpha-1}}{\gamma^{\alpha-1}\Gamma(\alpha)}\int_\rho^\gamma(\gamma-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta\\&\quad+\frac{1}{\Gamma(\alpha)}\int_\rho^t(t-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta\\&=\gamma^{2-\alpha}t^{\alpha-1}+\int_\rho^\gamma G(t,\theta)f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta,\end{aligned} \tag{16}$$

where

$$G(t,\theta)=\frac{1}{\Gamma(\alpha)}\begin{cases}(t-\theta)^{\alpha-1}-\left(\frac{t}{\gamma}\right)^{\alpha-1}\cdot(\gamma-\theta)^{\alpha-1}, & \rho\le\theta\le t\le\gamma\\-\left(\frac{t}{\gamma}\right)^{\alpha-1}\cdot(\gamma-\theta)^{\alpha-1}, & \rho\le t\le\theta\le\gamma\end{cases} \tag{17}$$

Define the operator $T_1:C[0,T]\to C[0,T]$, by $T_1x(t):=\gamma^{2-\alpha}t^{\alpha-1}+\int_\rho^\gamma G(t,\theta)f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta$, then we have

$$\begin{aligned}|T_1x(t)|&=\left|\gamma^{2-\alpha}t^{\alpha-1}+\int_\rho^\gamma G(t,\theta)f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta\right|\\&=\left|\gamma^{2-\alpha}t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_\rho^t(t-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta-\frac{t^{\alpha-1}}{\gamma^{\alpha-1}\Gamma(\alpha)}\int_\rho^\gamma(\gamma-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta\right|\\&\le\left|\gamma^{2-\alpha}t^{\alpha-1}\right|+\frac{1}{\Gamma(\alpha)}\int_\rho^t(t-\theta)^{\alpha-1}\left|f\left(\theta,x(\theta),x^{[2]}(\theta)\right)\right|d\theta\\&\quad+\frac{t^{\alpha-1}}{\gamma^{\alpha-1}\Gamma(\alpha)}\int_\rho^\gamma(\gamma-\theta)^{\alpha-1}\left|f\left(\theta,x(\theta),x^{[2]}(\theta)\right)\right|d\theta\\&\le\left|\gamma^{2-\alpha}t^{\alpha-1}\right|+\frac{Q}{\Gamma(\alpha)}\int_\rho^t(t-\theta)^{\alpha-1}d\theta+\frac{Kt^{\alpha-1}}{\gamma^{\alpha-1}\Gamma(\alpha)}\int_\rho^\gamma(\gamma-\theta)^{\alpha-1}d\theta\\&\le\left|\gamma^{2-\alpha}\lambda_1\right|+\frac{\left(Q\gamma^{\alpha-1}+K\lambda_1\right)(\gamma-\rho)^\alpha}{\alpha\gamma^{\alpha-1}\Gamma(\alpha)},\end{aligned}$$

where $\lambda_1=\max\left\{\rho^{\alpha-1},\gamma^{\alpha-1}\right\}$, so we get the uniform boundedness of the function. Next we shall present the equicontinuity,

$$
\begin{aligned}
|T_1x(t+\varepsilon)-T_1x(t)| &= \gamma^{2-\alpha}(t+\varepsilon)^{\alpha-1}-\gamma^{2-\alpha}t^{\alpha-1}-\frac{(t+\varepsilon)^{\alpha-1}}{\gamma^{\alpha-1}\Gamma(\alpha)}\int_\rho^\gamma(\gamma-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta \\
&+\frac{t^{\alpha-1}}{\gamma^{\alpha-1}\Gamma(\alpha)}\int_\rho^\gamma(\gamma-s)\theta^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta+\frac{1}{\Gamma(\alpha)}\int_\rho^{t+\varepsilon}(t+\varepsilon-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta \\
&-\frac{1}{\Gamma(\alpha)}\int_\rho^t(t-\sigma)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta \\
&=\gamma^{2-\alpha}[(t+\varepsilon)^{\alpha-1}-t^{\alpha-1}]+\frac{t^{\alpha-1}-(t+\varepsilon)^{\alpha-1}}{\gamma^{\alpha-1}\Gamma(\alpha)}\int_\rho^\gamma(\gamma-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta \\
&+\frac{1}{\Gamma(\alpha)}\int_t^{t+\varepsilon}(t+\varepsilon-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta+\frac{1}{\Gamma(\alpha)}\int_\rho^t\left[(t+\varepsilon-\theta)^{\alpha-1}-(t-\theta)^{\alpha-1}\right]f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta \\
&\le\gamma^{2-\alpha}[(t+\varepsilon)^{\alpha-1}-t^{\alpha-1}]+\frac{Q\left[(t+\varepsilon)^{\alpha-1}-t^{\alpha-1}\right]}{\gamma^{\alpha-1}\Gamma(\alpha)}\cdot(\gamma-\rho)^\alpha-\frac{Q\varepsilon^\alpha}{\alpha\Gamma(\alpha)}+\frac{Q}{\alpha\Gamma(\alpha)}\left[\varepsilon^\alpha-(t+\varepsilon-\rho)^\alpha+(t-\rho)^\alpha\right] \\
&=\gamma^{2-\alpha}\left[t^{\alpha-1}-(t+\varepsilon)^{\alpha-1}\right]+\frac{Q(\gamma-\rho)^\alpha}{\alpha\gamma^{\alpha-1}\Gamma(\alpha)}\left[(t+\varepsilon)^{\alpha-1}-t^{\alpha-1}\right]+\frac{Q}{\alpha\Gamma(\alpha)}\left[(t-\rho)^\alpha-(t+\varepsilon-\rho)^\alpha\right],
\end{aligned}
$$

it is easy to see that if $\varepsilon\to 0$, then $|T_1x(t+\varepsilon)-T_1x(t)|\to 0$, such that T_1 is equicontinuous, thus there has a solution of (2).

Since (H1), (H2) are satisfied, then we will prove that there has a unique solution of (2). According to the assumption (H1), one can obtain that

$$
\begin{aligned}
\frac{d(T_1x(t))}{dt} &= (\alpha-1)\gamma^{2-\alpha}t^{\alpha-1}-\frac{(\alpha-1)t^{\alpha-1}}{\gamma^{\alpha-1}\Gamma(\alpha)}\int_\rho^\gamma(\gamma-\theta)^{\alpha-1}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta \\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_\rho^t(t-\theta)^{\alpha-2}f\left(\theta,x(\theta),x^{[2]}(\theta)\right)d\theta \\
&\le(\alpha-1)\gamma^{2-\alpha}\lambda_2-\frac{K(\alpha-1)\lambda_3(\gamma-\rho)^\alpha+\alpha\gamma^{\alpha-1}Q(\gamma-\rho)^{\alpha-1}}{\alpha\gamma^{\alpha-1}\Gamma(\alpha)},
\end{aligned}
$$

where $\lambda_2=\max\{\rho^{\alpha-2},\gamma^{\alpha-2}\}$, $\lambda_3=\min\{\rho^{\alpha-2},\gamma^{\alpha-2}\}$, which implies that T_1 is a bounded operator. If we assume that there exists two fixed points such as $\chi_1(t),\chi_2(t)$, and $\chi_1(t)\ne\chi_2(t)$, where

$$\chi_1,\chi_2\in C^{n-1}[\rho,\gamma],$$

we have

$$(\chi_1(t)-\chi_2(t))D^\alpha(\chi_1(t)-\chi_2(t))=\left(f\left(t,\chi_1(t),\chi_1^{[2]}(t)\right)-f\left(t,\chi_2(t),\chi_2^{[2]}(t)\right)\right)(\chi_1(t)-\chi_2(t)),$$

such that,

$$
\begin{aligned}
D^\alpha|\chi_1(t)-\chi_2(t)|^2 &\le\frac{1}{2}(\chi_1(t)-\chi_2(t))D^\alpha(\chi_1(t)-\chi_2(t)) \\
&\le\frac{1}{2}\left(f\left(t,\chi_1(t),\chi_1^{[2]}(t)\right)-f\left(t,\chi_2(t),\chi_2^{[2]}(t)\right)\right)(\chi_1(t)-\chi_2(t)),
\end{aligned}
$$

integrating the both sides and according to (H2), then we have

$$
\begin{aligned}
&|\chi_1(t)-\chi_2(t)|^2 \\
&\le\frac{1}{2}\int_\rho^\gamma G(t,\theta)\left(f\left(\theta,\chi_1(\theta),\chi_1^{[2]}(\theta)\right)-f\left(\theta,\chi_2(\theta),\chi_2^{[2]}(\theta)\right)\right)(\chi_1(\theta)-\chi_2(\theta))d\theta \\
&\le\frac{(M+SF)}{2}\int_\rho^\gamma G(t,\theta)|\chi_1(t)-\chi_2(t)|^2d\theta,
\end{aligned}
$$

where $S=(\alpha-1)\gamma^{2\text{-}\alpha}\lambda_2-\dfrac{N(\alpha-1)\lambda_3(\gamma-\rho)^\alpha+\alpha\gamma^{\alpha-1}Q(\gamma-\rho)^{\alpha-1}}{\alpha\gamma^{\alpha-1}\Gamma(\alpha)}$, in addition of the Lemma 1.1, we have $\dfrac{(M+SF)}{2}\int_\rho^\gamma G(t,\theta)|\chi_1(\theta)-\chi_2(\theta)|^2d\theta\le 0$, such that $|\chi_1(t)-\chi_2(t)|^2\le 0$, it is obvious that the conclusion contradicts the hypothesis, so the uniqueness is proved. The proof is completed.

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References

[1]. Y. Cheng, S. Gao, and Y. Wu. Exact controllability of fractional order evolution equations in banach spaces. Advances in Difference Equations, 2018(1):332, 2018.

[2]. Y. Cui. Uniqueness of solution for boundary value problems for fractional differential equations. Applied Mathematics Letters, 51:48–54, 2016.

[3]. P. W. Eloe and T. Masthay. Initial value problems for caputo fractional differential equations. Journal of Fractional Calculus and Applications, 9(2), 2018.

[4]. E. R. Kaufmann. Existence and uniqueness of solutions for a second-order iterative boundary-value problem. Electronic Journal of Differential Equations, 2018(150):1–6, 2018.

[5]. W. Ke. On the equation $x'(t)=f(x(x(t)))$. Funkcialaj Ekvacioj, 33(3), 1990.

[6]. V. R. Petukhov. On a boundary value problem. Trudy sem. teorii diff. uravn. otklon. arg, 3:252–255, 1965.

[7]. H. Ye, J. Gao, and Y. Ding. A generalized gronwall inequality and its application to a fractional differential equation. Journal of Mathematical Analysis & Applications, 328(2):1075–1081.

[8]. X. Zhang, L. Liu, and Y. Wu. Multiple positive solutions of a singular fractional differential equation with negatively perturbed term. Mathematical and Computer Modelling, 55:1263–1274, 2012.